Published by Institute of Physics Publishing for SISSA

Received: *August 29, 2005*
Accepted: *September 30, 2005*
Published: *October 20, 2005*

PP-wave light-cone free string field theory at finite temperature

Maria Cristina B. Abdalla, Alexandre L. Gadelha and Daniel L. Nedel

Instituto de Física Teórica, Unesp, Pamplona
145, São Paulo, SP, 01405-900, Brazil
E-mail: mabdalla@ift.unesp.br, gadelha@ift.unesp.br, daniel@ift.unesp.br

Abstract: In this paper, a real-time formulation of light-cone pp-wave string field theory at finite temperature is presented. This is achieved by developing the thermo field dynamics (TFD) formalism in a second quantized string scenario. The equilibrium thermodynamic quantities for a pp-wave ideal string gas are derived directly from expectation values on the second quantized string thermal vacuum. Also, we derive the real-time thermal pp-wave closed string propagator. In the flat space limit it is shown that this propagator can be written in terms of Theta functions, exactly as the zero temperature one. At the end, we show how superstrings interactions can be introduced, making this approach suitable to study the BMN dictionary at finite temperature.

Keywords: Thermal Field Theory, String Field Theory, Penrose limit and pp-wave background.

Contents

1. Introduction **1**

2. Free string field theory **3**

3. Finite temperature system **6**
- 3.1 The duplication of degrees of freedom 6
- 3.2 Thermal vacuum and thermal operators 8
- 3.3 The entropy operator 9
- 3.4 The thermal energy and the free energy 11

4. The thermal propagator **13**

5. Conclusions and discussions **17**

1. Introduction

The formulation of superstring theory at finite temperature is fundamental to many applications, such as early string cosmology and black hole physics. One of the fascinating features of string theory at finite temperature is the exponential growth of states as function of energy. Such behavior leads to a temperature above which the partition function diverges (the Hagedorn temperature). If the Hagedorn temperature denotes a phase transition, the true degrees of freedom of the theory at high energy may be others than those of the perturbative string. However, in spite of many works about finite temperature string theory,[1] a precise understanding of the Hagedorn temperature and the true degrees of freedom at higher temperatures is still lacking. In this way, beyond the applications, finite temperature studies can provide fundamental issues of string theory itself.

Recently, the study of the finite temperature string theory in a pp-wave background has provided great advances in this direction [2 − 9]. In the context of the plane wave limit of the ADS/CFT correspondence, the BMN correspondence [10 − 12], it is possible to assert a dual description of the Hagedorn temperature in terms of a Yang-Mills temperature [2]. In this scenario the Hagedorn behavior can be related to deconfinement/confinement phase transition on the gauge side [13, 14]. However, in those works superstring interactions were neglected. As shown in [15 − 17], the energy density of states at high temperature favours the formation of a single long string which carries most of the available energy. In the thermodynamic limit this long string has numerous opportunities to intersect itself [18]. So, the physical picture of an ideal string gas at high temperature is suspect and it is fundamental to have a finite temperature description of the superstring interactions, in order

[1] See for example [1] and references therein.

to clarify what is the physics at the Hagedorn limit and keep the ideal gas approximation under control.

The recent development of the plane wave limit of the ADS/CFT correspondence brought to light an ancient string theory subject: the Light-Cone String Field Theory (LCSFT) [19 – 22]. The development of the LCSFT to study pp-wave superstring is fundamental to understand the BMN dictionary, because it naturally connects both sides of the correspondence when higher genus corrections in the plane wave spectrum are taken into account. Considering that the LCSFT is the main tool to understand superstring interactions in a pp-wave background [23 – 26], a finite temperature description of this theory could help to understand many aspects of string thermodynamics that still puzzle us. In general, as LCSFT is a multi-strings theory, its finite temperature version may be the natural scenario to study string thermodynamics in the light-cone.

In order to construct the pp-wave LCSFT at finite temperature it is interesting to use a formalism that keeps intact all the operator machinery of the theory at zero temperature. Therefore, in this work we have further developed the Thermo Field Dynamics (TFD) [27] to introduce temperature in LCSFT. There are many characteristics of TFD that make this formalism suitable to study string field theory at finite temperature. Let's list some of them.

- TFD is a canonical and an operator approach. The statistical average of an observable is derived from an expectation value in a pure state (the thermal vacuum), that depends on temperature. In this way, the usual quantum mechanical perturbation theory that is used in LCSFT can be used at finite temperature by means of this formalism. So, by developing the TFD in the LCSFT context, we get a well-suited tool to study string interactions at finite temperature.

- One of the TFD main objects is the entropy operator. This operator has been used, for instance, to calculate entanglement entropy of black holes [28], and it can be a powerful tool to study black hole entropy from the string point of view.

- TFD has a BRST formulation [29]. Then, it can be developed further to construct a covariant string field theory at finite temperature. In particular, in [30] the TFD was used for this purpose, but it was employed within a path integral formulation, different from the canonical operator approach used here, as well as in [29].

- TFD shares common characteristics with the C^* algebra approach to statistical mechanics [31, 32]. In fact, it was shown by Ojima [29] the equivalence of TFD with the Haag-Hugenholtz-Winnink (HHW) algebraic formulation. Therefore, TFD can bring interesting mathematical questions involving axiomatic statistical mechanics to the string field theory scenario.

- TFD is a real-time formalism: it can be used to study time dependent processes as dissipations and time-dependent backgrounds. Owing to the evolution of particle distribution, a real-time formalism seems to be the appropriated one for theories containing gravity [33].

JHEP10(2005)063

In this work we construct all the TFD ingredients in the LCSFT context, in order to have tools to explore multi-string effects at finite temperature, which include superstring interactions, multi-string bound states formation as well as the BMN dictionary at finite temperature. We concentrate here on the free theory and the interactions shall be introduced in a future work. The finite temperature formulation of LCSFT presented here is quite general, allowing to take into account time dependent process and out of equilibrium thermodynamics effects. We construct the second quantized thermal vacuum for pp-wave string theory and show that this state is an entanglement of strings. In the equilibrium, we compute thermodynamic quantities of an ideal string gas directly from expectation values on the second quantized string thermal vacuum. Also, we derive the real-time thermal propagator for pp-wave closed string. We show that in the flat space limit, the thermal propagator can be written in terms of Theta functions, exactly as the zero temperature one.

This work is divided as follows. In section 2 we introduce the basic elements of LCSFT necessary to apply TFD. In section 3 the system is led to finite temperature and the TFD approach is applied in the context of free LCSFT. The entropy operator is introduced and the entropy as well as the thermal energy are obtained. Closing the section we obtain the light-cone thermal distribution for a pp-wave string gas and derive the free energy. Section 4 is deserved to derive the pp-wave thermal string propagator in a flat and in a pp-wave background. At the end, in the section 5 we discuss the relationship between first and second quantization of string at finite temperature in the TFD approach. The effect of the thermal Bogoliubov transformation in both cases and the respective topological interpretation are presented. Also we show how to introduce string interactions using TFD quantum mechanical perturbation theory.

2. Free string field theory

In this section the basic concepts of the free light-cone SFT in a pp-wave background are introduced. Following reference [23], the bosonic part of the light-cone action for first quantized closed pp-wave superstring is:

$$S = \frac{e(p^+)}{8\pi} \int d\tau \int_0^{4\pi|p^+|} d\sigma \left(\partial_+ x^i \partial_- x^j - \mu^2 x^i x^j\right) \delta_{ij} \,, \tag{2.1}$$

where we have set $\alpha' = 2$, $e(p^+)$ is the signal of p^+, $\partial_\pm = \partial_\tau \pm \partial_\sigma$ and $i, j = 0, \ldots, D-2$ labels the transverse directions in the light-cone. The mode expansions for the string coordinate $x^i(\sigma)$ and density momentum $p^i(\sigma)$ are:

$$x^i(\sigma) = x_0^i + \frac{1}{\sqrt{2}} \sum_{k\neq 0} \left(x^i_{|k|} - ie(k) x^i_{-|k|}\right) e^{i\frac{k\sigma}{2|p^+|}} \,, \tag{2.2}$$

$$p^i(\sigma) = \frac{1}{4\pi p^+} \left[p_0^i + \sum_{k\neq 0} \left(p^i_{|k|} - ie(k) p^i_{-|k|}\right) e^{i\frac{k\sigma}{2|p^+|}}\right] \,, \tag{2.3}$$

where

$$x_k^i - i x_{-k}^i = \sqrt{\frac{2}{\omega_k}} \left(\bar{\alpha}_k^i + \alpha_k^{i\dagger}\right) , \qquad p_k^i - ie(k) p_{-k}^i = \sqrt{\frac{2}{\omega_k}} \left(\bar{\alpha}_k^i - \alpha_k^{i\dagger}\right) , \tag{2.4}$$

for

$$\omega_k = \sqrt{k^2 + (2p^+\mu)^2}\,. \tag{2.5}$$

Using x_k^i and p_k^i the world-sheet hamiltonian can be written as:

$$h = \frac{1}{2p^+} \sum_{k=-\infty}^{+\infty} \left[p_k^2 + \frac{1}{4}\omega_k^2 x_k^2 \right]. \tag{2.6}$$

As usual in the light-cone SFT these modes allow us to define a new creation-annihilation basis, whose indices range from $-\infty$ to $+\infty$:

$$a_k^i = \frac{1}{\sqrt{\omega_k}}\, p_k^i - \frac{i}{2}\frac{1}{\sqrt{\omega_k}}\, x_k^i\,, \qquad a_k^{i\,\dagger} = \frac{1}{\sqrt{\omega_k}}\, p_k^i + \frac{i}{2}\frac{1}{\sqrt{\omega_k}}\, x_k^i\,, \tag{2.7}$$

with $[a_k^i, a_m^{j\,\dagger}] = \delta^{ij}\delta_{km}$. In this new basis the hamiltonian is:

$$h = \frac{1}{2p+}\left[\sum_{k=-\infty}^{\infty} \omega_k\, a_k^{i\,\dagger} a^i + A \right], \tag{2.8}$$

where A is the normal ordering constant:

$$A = \frac{\delta^i{}_i}{2} \sum_{k=-\infty}^{\infty} \omega_k\,. \tag{2.9}$$

The first quantized Fock space is constructed by acting with the creation operators $a_k^{i\,\dagger}$ on the vacuum $|0\rangle$, which is annihilated by a_k^i. As we are working with closed strings, the physical states satisfy the level matching condition:

$$\sum_{i=1}^{d-2} \sum_{k=-\infty}^{\infty} k\, n_k^i\, |\psi\rangle = 0\,, \tag{2.10}$$

where n_k^i is the eigenvalue of the number operator $N_k^i = a_k^{i\,\dagger} a_k^i$.

Let us now define the fundamental object of the LCSFT, the string field $\Phi[x^i(\sigma), x^-, x^+]$. The string field is a functional, associating a collection of real numbers with each curve $x^i(\sigma)$ in the transverse light-cone space. It is an operator defined in the multi-string Hilbert space $\mathcal{H} = |vacuum\rangle \oplus_{m=1}^{\infty} \mathcal{H}_m$, where the m string Hilbert space, $\mathcal{H}_m$, is a direct product of m single Hilbert spaces $\mathcal{H}_1$. The string field also satisfies the following equal time commutation relation

$$\Big[\Phi[x^i(\sigma), x^-, x^+], \Phi^\dagger[y^i(\sigma), y^-, x^+]\Big] = \delta(x^- - y^-)\delta^{d-2}(x^i - y^i)\,. \tag{2.11}$$

As usual in light-cone field theories, the dynamics is dictated by the Schrödinger equation. For the string field, it is a functional Schrödinger equation, defined in the light-cone configuration space by:

$$i\partial_+ \Phi[x^i(\sigma), x^-, x^+] = \frac{1}{2} \sum_{k=-\infty}^{\infty} \left[-\frac{\partial^2}{\partial x_k^{i\,2}} + \omega_k^2 (x_k^i)^2 \right] \Phi[x^i(\sigma), x^-, x^+]\,. \tag{2.12}$$

The solution of this equation will be useful in section 4 to derive the real-time thermal propagator in configuration space. It is given by

$$\Phi\left[x^i(\sigma), x^-, x^+\right] = \int dp^+ \sum_{\{n^i_k\}} A_{\{n^{(i)}_k\}}(p^+) e^{-i\left(x^+p^- + x^-p^+\right)} \prod_{k=-\infty}^{\infty} \phi_{\{n^{(i)}_k\}}\left(x^i_k\right) + \text{h.c.}\,, \quad (2.13)$$

where $\phi_{\{n^{(i)}_k\}}\left(x^i_k\right)$ is written in terms of Hermite polynomials:

$$\phi_{\{n^{(i)}_k\}}\left(x^i_k\right) = \prod_{i=1}^{D-2} H_{\{n^i_k\}}(\sqrt{\omega_k} x^i_k) e^{-\omega_k \frac{(x^i_k)^2}{2}} \sqrt{\frac{\sqrt{\omega_k}}{\sqrt{\pi} 2^{n^i_k} (n^i_k!)}}\,. \quad (2.14)$$

The sum over $\{n^{(i)}_k\}$ means the sum over all possible sets of occupation numbers taking into account the level matching condition (2.10). $A_{\{n^{(i)}_k\}}(p^+)$ creates and destroys an entire string with occupation indices $\{n^{(i)}_k\}$ and fixed p^+. Thus, $A_{\{n^{(i)}_k\}}(p^+)$ is the operator that leads the m-string Hilbert space $\mathcal{H}_m$ into $\mathcal{H}_{m\pm 1}$ with plus sign when $p^+ < 0$ and the minus one when $p^+ > 0$. Writing $A^\dagger_{\{m^{(j)}_k\}}(p^+) = A_{\{n^{(i)}_k\}}(-p^+)$ for $p^+ > 0$, the commutation relation for the second quantized creation and annihilation operators is

$$\left[A_{\{n^{(i)}_k\}}(p^+), A^\dagger_{\{m^{(j)}_k\}}(q^+)\right] = \delta(p^+ - q^+)\, \delta_{\{n^{(i)}_k\},\{m^{(j)}_k\}}\,, \quad (2.15)$$

and the vacuum of the second quantized vector space is defined as usual:

$$A_{\{n^{(i)}_k\}}(p^+)\,|0) = 0\,. \quad (2.16)$$

It is convenient for our purpose to work in the p^+ momentum space and expand the string field $\Phi(p^+)$ in terms of the eigenstates of the string world-sheet number operators, writing [34]

$$\Phi(p^+) = \frac{1}{\sqrt{|p^+|}} \sum_{\{n^{(i)}_k\}} \left|\{n^{(i)}_k\}\,, p^+\right\rangle A_{\{n^{(i)}_k\}}(p^+)\,, \quad (2.17)$$

where the ket $|\{n^{(i)}_k\}, p^+\rangle$ is a general string state in the first quantized string space. Note that $\Phi(x, p^+) = \langle x|\,\Phi(p^+)$. From the above expression it is clear that the string field is a state in the first quantized closed string Fock space, $\mathcal{F}$, and an operator in the second quantized space, $\mathcal{H}$. Let's close this section writing the second quantized string Hamlitonian. In general, the second quantized generators are derived from the first quantized ones, by means of the map

$$\mathcal{G} = \int dp^+ \mathcal{D}p(\sigma)\, p^+ \Phi^\dagger[p^i(\sigma), p^+]\, g\, \Phi[p^i(\sigma), p^+]\,, \quad (2.18)$$

which gives the second quantized hamiltonian:

$$H = \int dp^+\, p^+ \Phi^\dagger(p^+)\, h\, \Phi(p^+) = \int dp^+ \sum_{\{n^{(i)}_k\}} E_{\{n^{(i)}_k\}} A^\dagger_{\{n^{(i)}_k\}}(p^+) A_{\{n^{(i)}_k\}}(p^+)\,, \quad (2.19)$$

JHEP10(2005)063

with

$$E_{\{n_k^{(i)}\}} = \frac{1}{2p^+}\left[\sum_{k=-\infty}^{\infty} w_k n_k^i + A\right], \tag{2.20}$$

where A is defined in (2.9).

With the ingredients of the free light-cone string field theory in the pp-wave background here presented, we are able to construct its finite temperature analogous and obtain the free string gas for this system.

3. Finite temperature system

TFD, when proposed by Takahashi and Umezawa [27], was based on the idea of interpreting the statistical average of an operator, $\mathcal{O}$, as its expectation value in a temperature dependent state called thermal vacuum:

$$\langle \mathcal{O} \rangle \equiv \frac{\mathrm{Tr}\left[\mathcal{O}\, e^{\beta H}\right]}{\mathrm{Tr}\left[e^{\beta H}\right]} \equiv \langle\, 0\,(\beta)\,|\,\mathcal{O}\,|\,0\,(\beta)\,\rangle\,, \tag{3.1}$$

where β is the inverse of the temperature and H represents the hamiltonian operator of the system under consideration. Basically two implications lie behind such a proposal. Firstly, it is necessary a duplication of the system's degrees of freedom. Secondly, the use of an specific Bogoliubov transformation, the so-called thermal Bogoliubov transformation, to lead the system to a finite temperature. The implementation of the doubling, as well as the specifications for the thermal Bogoliubov transformation, will be presented in the following subsections, in the context of the system approached in this paper. This section as a whole has the aim to introduce the basic elements of TFD necessary to obtain the free light-cone SFT at finite temperature, with the perspective of developing a TFD construction to study string field interactions at finite temperature. Here we derive the Bose multi-string distributions and the free energy for the ideal string gas in the pp-wave background. Entropy and thermal energy are also calculated from the thermal expectation values.

3.1 The duplication of degrees of freedom

As it was mentioned above, the TFD approach consists of a doubling of the system's degrees of freedom followed by a thermal Bogoliubov transformation. Following the TFD algorithm the duplication is implemented by introducing a copy of the original Hilbert space of the system $\mathcal{H}$. Denoting the copy as $\widetilde{\mathcal{H}}$, the total Hilbert space that take place for the doubled system is $\widehat{\mathcal{H}} = \mathcal{H} \otimes \widetilde{\mathcal{H}}$. Here and in the following, we refer to the copy of the original system as tilde system or auxiliary system. There is a map between both systems given by the tilde conjugation rules

$$(AB)^{\sim} = \widetilde{A}\widetilde{B}\,, \tag{3.2}$$

$$(c_1 A + c_2 B)^{\sim} = \left(c_1^* \widetilde{A} + c_2^* \widetilde{B}\right), \tag{3.3}$$

$$\left(A^\dagger\right)^{\sim} = \widetilde{A}^\dagger\,, \tag{3.4}$$

$$\left(\widetilde{A}\right)^{\sim} = A\,, \tag{3.5}$$
$$\left(|0\,(\theta)\rangle\right)^{\sim} = |0\,(\theta)\rangle\,, \tag{3.6}$$
$$\left(\langle 0\,(\beta)|\right)^{\sim} = \langle 0\,(\beta)|\,, \tag{3.7}$$

with A and B representing bosonic operators and c_1, $c_2 \in \mathbb{C}$.

As the string field is a state in $\mathcal{F}$, the doubling of the degrees of freedom implies a duplication of the basis states. Defining the duplicated number basis by:

$$\left|\,\{n_k^{(i)}\}\,\right\rangle\!\!\right\rangle = \left|\{n_k^{(i)}\},p^+\right\rangle \otimes \left|\widetilde{\{n_k^{(i)}\},p^+}\right\rangle, \tag{3.8}$$

the string and tilde string fields have the following expansion in the total Hilbert space:

$$\begin{aligned}
\mathbf{\Phi}(p^+) &= \frac{1}{\sqrt{|p^+|}} \sum_{\{n_k^{(i)}\}} \left|\,\{n_k^{(i)}\},p^+\right\rangle\!\!\right\rangle A_{\{n_k^{(i)}\}}(p^+)\,,\\
\widetilde{\mathbf{\Phi}}(p^+) &= \frac{1}{\sqrt{|p^+|}} \sum_{\{n_k^{(i)}\}} \left|\,\{n_k^{(i)}\},p^+\right\rangle\!\!\right\rangle \widetilde{A}_{\{n_k^{(i)}\}}(p^+)\,,
\end{aligned} \tag{3.9}$$

where the tilde field expansion is obtained just using the tilde conjugation rules. Also, the tilde conjugation rules provide that the dynamics of the auxiliar system is the same as the original one. At this point it is interesting to emphasize that the aplication of the TFD algorithm in second quantized string demands a duplication of both, the first quantized Hilbert space $\mathcal{F}$ and the second quantized Hilbert space $\mathcal{H}$. In the string first quantized TFD aplications, the auxiliary system is interpreted as an auxiliary string [56]. So, the field string is now a functional of two independent strings. This fact gives rise to an interesting topological interpretation, to be discussed in the last section.

The canonical commutation relations for the tilde string creation and annihilation operators are the same as the original ones given at (2.15), and elements from different spaces commute. The vacuum of the total system is now defined by

$$A_{\{n_k^{(i)}\}}(p^+)\;|0\rangle\!\rangle = \widetilde{A}_{\{n_k^{(i)}\}}(p^+)\;|0\rangle\!\rangle = 0\,. \tag{3.10}$$

Finally, let's define the the total second quantized hamiltonian. The Heisenberg equations for the Heisenberg original and tilde fields, together with the tilde conjugation rules, demand that the total second quantized hamiltonian is not H anymore, but it is given by

$$\widehat{H} = H - \widetilde{H}\,, \tag{3.11}$$

where H is defined in (2.19) and $\widetilde{H}$ is obtained by using the tilde conjugation rules. This hamiltonian structure is usual in the TFD approach and it is very important to define an interaction representation at finite temperature, which allows to use the same quantum mechanical perturbation theory defined at zero temperature.

3.2 Thermal vacuum and thermal operators

Once we have defined the doubled system, we are ready to introduce the thermal Bogoliubov transformation. It is such that its generator, $\mathbf{G}$, consists of bilinear terms composed by creation and annihilation operators from both, original and tilde systems. Besides this, it must satisfy what Umezawa called $\mathbf{G}$-symmetry of TFD [42], defined by the following three requirements, generalized here to string field theory: the generator must induce a transformation of the form

$$\begin{pmatrix} A_{\{n_k^{(i)}\}}(\theta) \\ \widetilde{A}^\dagger_{\{n_k^{(i)}\}}(\theta) \end{pmatrix} = e^{-i\mathbf{G}} \begin{pmatrix} A_{\{n_k^{(i)}\}} \\ \widetilde{A}^\dagger_{\{n_k^{(i)}\}} \end{pmatrix} e^{i\mathbf{G}} = \mathbb{B}_{\{n_k^{(i)}\}} \begin{pmatrix} A_{\{n_k^{(i)}\}} \\ \widetilde{A}^\dagger_{\{n_k^{(i)}\}} \end{pmatrix}, \tag{3.12}$$

$$\begin{pmatrix} A^\dagger_{\{n_k^{(i)}\}}(\theta) & -\widetilde{A}_{\{n_k^{(i)}\}}(\theta) \end{pmatrix} = \begin{pmatrix} A^\dagger_{\{n_k^{(i)}\}} & -\widetilde{A}_{\{n_k^{(i)}\}} \end{pmatrix} \mathbb{B}^{-1}_{\{n_k^{(i)}\}}, \tag{3.13}$$

where by θ we denote the transformation parameter that encodes the temperature and further relevant theory parameters dependences. $\mathbb{B}$ is the transformation matrix with the following general form

$$\mathbb{B}_{\{n_k^{(i)}\}} = \begin{pmatrix} u_{\{n_k^{(i)}\}} & v_{\{n_k^{(i)}\}} \\ v^*_{\{n_k^{(i)}\}} & u^*_{\{n_k^{(i)}\}} \end{pmatrix}, \tag{3.14}$$

with $u_{\{n_k^{(i)}\}} u^*_{\{n_k^{(i)}\}} - v_{\{n_k^{(i)}\}} v^*_{\{n_k^{(i)}\}} = 1$ for bosonic systems in order to the transformation be canonical. The generator must commute with the total hamiltonian of the system (3.11) and finally, the generator must change its sign under tilde conjugation rules implying thermal vacuum invariance under tilde conjugation.

The general TFD structure presented above allows us to observe that more than one generator satisfy such a requirements. In fact, TFD can be constructed taking into account a general generator that is a linear combination of the possible choices of $\mathbf{G}$ [42 – 50]. Here we will restrict the TFD construction to one generator only, following the originally proposed by Takahashi and Umezawa

$$\mathbf{G} = -i \sum_{\{n_k^{(i)}\}} \theta_{\{n_k^{(i)}\}} \left(A_{\{n_k^{(i)}\}} \widetilde{A}_{\{n_k^{(i)}\}} - A^\dagger_{\{n_k^{(i)}\}} \widetilde{A}^\dagger_{\{n_k^{(i)}\}} \right). \tag{3.15}$$

It satisfies the $\mathbf{G}$-symmetry and the thermal Bogoliubov matrix elements presented in (3.14) are explicitly written as

$$u_{\{n_k^{(i)}\}} = \cosh(\theta_{\{n_k^{(i)}\}}), \qquad v_{\{n_k^{(i)}\}} = -\sinh(\theta_{\{n_k^{(i)}\}}). \tag{3.16}$$

The second quantized thermal vacuum is defined by

$$A_{\{n_k^{(i)}\}}(\theta)\, |0(\theta)) = \widetilde{A}_{\{n_k^{(i)}\}}(\theta)\, |0(\theta)) = 0, \tag{3.17}$$

where the thermal annihilation operators were obtained from expression (3.13). The second quantized thermal space is constructed by cyclic applications of the thermal string creation operators $A^{\dagger}_{\{n_k^{(i)}\}}(\theta)$ and $\widetilde{A}^{\dagger}_{\{n_k^{(i)}\}}(\theta)$. The above definition for the thermal vacuum gives rise to the so-called thermal state condition

$$\left[A_{\{n_k^{(i)}\}} - \tanh(\theta_{\{n_k^{(i)}\}})\widetilde{A}^{\dagger}_{\{n_k^{(i)}\}}\right] |0(\theta)\,\rangle = 0\,. \tag{3.18}$$

From (3.18) it is possible to derive the KMS condition to be presented later. Also, it tells us that the annihilation operator of the original system does not annihilate the transformed vacuum. In fact, the thermal vacuum satisfying expression (3.17) is explicitly written as

$$|0(\theta)\,\rangle = e^{-i\mathbf{G}}\ |0\,\rangle\!\rangle = \prod_{\{n_k^{(i)}\}} \left(\frac{1}{\cosh(\theta_{\{n_k^{(i)}\}})}\right) e^{\tanh(\theta_{\{n_k^{(i)}\}})A^{\dagger}_{\{n_k^{(i)}\}}(p^+)\widetilde{A}^{\dagger}_{\{n_k^{(i)}\}}(p^+)}\ |0\,\rangle\!\rangle\,, \tag{3.19}$$

with doubled vacuum defined in (3.10). The above structure manifests the thermal vacuum invariance under tilde conjugation and shows us that the second quantized thermal vacuum is a condensed state of string thermal pairs $A^{\dagger}_{\{n_k^{(i)}\}}(p^+)\,\widetilde{A}^{\dagger}_{\{n_k^{(i)}\}}(p^+)$.

3.3 The entropy operator

The aim of this section is to introduce the TFD entropy operator presenting some properties of it. The name of such an operator will be clear at the end of this subsection.

In our extension of TFD for string field theory the second quantized entropy operator is defined as

$$\widehat{K} = K - \widetilde{K}\,, \tag{3.20}$$

where

$$\begin{aligned} K = -\sum_{\{n_k^{(i)}\}} \Big[& A^{\dagger}_{\{n_k^{(i)}\}}(p^+)A_{\{n_k^{(i)}\}}(p^+)\ln\sinh^2\left(\theta_{\{n_k^{(i)}\}}\right) - \\ & - A_{\{n_k^{(i)}\}}(p^+)A^{\dagger}_{\{n_k^{(i)}\}}(p^+)\ln\cosh^2\left(\theta_{\{n_k^{(i)}\}}\right)\Big]\,, \end{aligned} \tag{3.21}$$

and $\widetilde{K}$ is obtained from K by tilde conjugation rules. The operator $\widehat{K}$ commutes with the generator $\mathbf{G}$ and the total hamiltonian $\widehat{H}$. An interesting property of the entropy operator is the fact that it can be used to lead the system from zero to a finite temperature, by considering the following expression for the thermal vacuum

$$|0(\theta)\,\rangle = e^{-\frac{1}{2}K} e^{\sum_{\{n_k^{(i)}\}} A^{\dagger}_{\{n_k^{(i)}\}}(p^+)\widetilde{A}^{\dagger}_{\{n_k^{(i)}\}}(p^+)}\ |0\,\rangle\!\rangle\,. \tag{3.22}$$

Using the relations

$$e^{-\frac{1}{2}K}\ |0\,\rangle\!\rangle = \prod_{\{n_k^{(i)}\}} \left(\frac{1}{\cosh(\theta_{\{n_k^{(i)}\}})}\right) |0\,\rangle\!\rangle\,, \tag{3.23}$$

$$e^{-\frac{1}{2}K}A^{\dagger}_{\{n_k^{(i)}\}}(p^+)\,e^{\frac{1}{2}K} = \tanh(\theta_{\{n_k^{(i)}\}})A^{\dagger}_{\{n_k^{(i)}\}}(p^+)\,, \tag{3.24}$$

JHEP10(2005)063

one finds the same thermal vacuum obtained from the thermal Bogoliubov generator, whose structure is presented in (3.19). $\widetilde{K}$ also furnishes the same thermal vacuum, making use of the tilde conjugation version of the above expressions.

One can go further in the use of the entropy operator to generate the thermal vacuum: if we define the following entangled state

$$
\begin{aligned}
|\mathcal{I}) &\equiv e^{\sum_{\{n_k^{(i)}\}} A^\dagger_{\{n_k^{(i)}\}}(p^+)\widetilde{A}^\dagger_{\{n_k^{(i)}\}}(p^+)} |0\rangle\rangle && (3.25)\\
&= \sum_{\{\mathbf{n_k^{(i)}}\}} \left|\{\mathbf{n_k^{(i)}}\}\right)\Big) && (3.26)\\
&= |0\rangle\rangle + \sum_{\{n_k^{(i)}\}} \left|\{n_k^{(i)}\}\right)\Big) + \sum_{\{n_k^{(i)}\}}\sum_{\{m_k^{(i)}\}} \left|\{n_k^{(i)}\},\{m_k^{(i)}\}\right)\Big) + \\
&\quad + \sum_{\{n_k^{(i)}\}}\sum_{\{m_k^{(i)}\}}\sum_{\{l_k^{(i)}\}} \left|\{n_k^{(i)}\},\{m_k^{(i)}\},\{l_k^{(i)}\}\right)\Big) + \cdots\,, && (3.27)
\end{aligned}
$$

the expression (3.22) can be written as

$$
|0(\theta)\,) = e^{-\frac{1}{2}K}\,|\mathcal{I}\,)\,, \tag{3.28}
$$

and by explicit calculation one finds

$$
|0(\theta)\,) = \sum_{\{n_k^{(i)}\}} \sqrt{W_{\{n_k^{(i)}\}}}\,|\mathcal{I}\,)\,, \tag{3.29}
$$

where

$$
W_{\{n_k^{(i)}\}} = \prod_{i=1}^{D-2}\prod_{k=-\infty}^{\infty} \frac{\sinh^{2n_k^i}(\theta_{\{n_k^i\}})}{\cosh^{2(n_k^i+1)}(\theta_{\{n_k^i\}})}\,, \tag{3.30}
$$

is related with the density matrix of the system.

The next property of the entropy operator is exactly what justifies its name: the thermal expectation value of the operator K defined in (3.21);

$$
\mathcal{S} = \int dp^+\,(\,0(\theta)|\,K\,|0(\theta)\,)\,. \tag{3.31}
$$

The strategy is to use the inverse of thermal Bogoliubov transformation given by (3.12) and (3.13) with matrix elements presented in (3.16), to write the second quantized creation and annihilation operators in terms of thermal operators. Applying them to the thermal vacuum defined in (3.17), results the following expression

$$
\mathcal{S} = -\int dp^+ \sum_{\{n_k^{(i)}\}} \left[\mathcal{N}_{\{n_k^{(i)}\}}(\theta)\ln\left(\mathcal{N}_{\{n_k^{(i)}\}}(\theta)\right) - \left(1+\mathcal{N}_{\{n_k^{(i)}\}}(\theta)\right)\ln\left(1+\mathcal{N}_{\{n_k^{(i)}\}}(\theta)\right)\right], \tag{3.32}
$$

where $\mathcal{N}_{\{n_k^{(i)}\}}(\theta)$ is the thermal expectation value of the second quantized number operator

$$
\mathcal{N}_{\{n_k^{(i)}\}}(\theta) = (\,0(\theta)\,|\,N(p^+)\,|0(\theta)\,) = \sinh^2\left(\theta_{\{n_k^{(i)}\}}\right), \tag{3.33}
$$

The p^+ dependence in (3.32) is implicit in the parameter θ as it will be clear in the next subsection where it will be shown that the expectation value of the second quantized number operator is in fact the thermal distribution of the system. Note that the expression (3.32) is formally identical to the general expression for the entropy of bosonic systems, explaining why the K operator is called the entropy operator.

One can proceed in a different way and perform the thermal expectation value of the entropy operator using the expression (3.29) for the thermal vacuum and hence obtaining the entropy of the system written in terms of the coefficients $W_{\{n_k^{(i)}\}}$:

$$\mathcal{S} = -\int dp^+ \sum_{\{n_k^{(i)}\}} W_{\{n_k^{(i)}\}} \ln W_{\{n_k^{(i)}\}} \,. \tag{3.34}$$

It is clear from equation (3.29) that the thermal vacuum generated by the Bogoliubov operator or by the entropy operator has an entanglement structure. Such an structure was explored in black hole physics [28], where the entanglement is dynamically generated by gravitational interactions. In this context, a TFD-like entropy operator was used to derive the entanglement black hole entropy and the result agree with the area law. In this scenario, it will be very interesting to use the entropy operator constructed here to compute black hole entropy from string entanglement.

3.4 The thermal energy and the free energy

As mentioned before, thermal quantities in TFD are obtained from the thermal expectation value of dynamical operators of the original (non tilde) system. As we are working in the light-cone gauge, in order to obtain the thermal energy of the system the second quantized version of the light-cone energy $p^0 = \frac{1}{\sqrt{2}}\,(p^- + p^+)$ is needed:

$$P^0 = \int dp^+\, p^+ \Phi^\dagger(p^+)\, p^0\, \Phi(p^+) = \frac{1}{\sqrt{2}} \int dp^+ \sum_{\{n_k^{(i)}\}} \left(E_{\{n_k^{(i)}\}} + p^+ \right) A^\dagger_{\{n_k^{(i)}\}}(p^+) A_{\{n_k^{(i)}\}}(p^+)\,, \tag{3.35}$$

where $E_{\{n_k^{(i)}\}}$ is defined in (2.20). The thermal energy of the system is the expectation value of the second quantized operator P^0. Denoting by $\mathcal{E}$ this energy, one has

$$\mathcal{E} = (\, 0(\theta)|\, P^0\, |0(\theta)\,) = \frac{1}{\sqrt{2}} \int dp^+ \sum_{\{n_k^{(i)}\}} \left(E_{\{n_k^{(i)}\}} + p^+ \right) \sinh^2\left(\theta_{\{n_k^{(i)}\}} \right) . \tag{3.36}$$

With the expression for the entropy and thermal energy of the second quantized system, let us note that such expressions are given in terms of the Bogoliubov matrix elements. Those thermal quantities we named in advance, once the thermal distribution of the system was not presented yet. In fact, the term thermal or temperature we have been using all along the paper was always in this sense. The TFD construction presented here is quite general in order to deal with time-dependent as well as dissipation phenomena. However our specific interest in this work is to deal with the equilibrium situation. Such a specification can be realized as follows. If $\mathcal{E}$ is the thermal energy and $\mathcal{S}$ the entropy, the free energy is defined as usual

$$\mathcal{F} = \mathcal{E} - T\mathcal{S}\,, \tag{3.37}$$

where T is identified as the temperature. Now, one can find that, the differentiation of the free energy with respect to the θ parameter is an extremum for

$$\mathcal{N}_{\{n_k^{(i)}\}} = \sinh^2(\theta_{\{n_k^{(i)}\}}) = \frac{e^{-\frac{\beta}{\sqrt{2}}(E_{\{n_k^{(i)}\}}+p^+)}}{1-e^{-\frac{\beta}{\sqrt{2}}(E_{\{n_k^{(i)}\}}+p^+)}}\,. \tag{3.38}$$

In this way, the Bose distribution for ideal string gas is derived. Replacing the above distribution in the expressions (3.32) and (3.36), the equation (3.37) for the free energy can be written as

$$\mathcal{F} = \frac{1}{\beta}\int dp^+ \sum_{\{n_k^{(i)}\}} \ln\left(1-e^{-\frac{\beta}{\sqrt{2}}(E_{\{n_k^{(i)}\}}+p^+)}\right). \tag{3.39}$$

In this expression, the sum over $\{n_k^{(i)}\}$ can be performed for each value of k, i, and must be constrained in order to take into account the level matching condition (2.10). This is usually made by means of a Lagrange multiplier τ_1

$$\sum_{\{n_k^{(i)}\}} \to \int_{-1/2}^{1/2} d\tau_1 \sum_{n_k^i} \prod_{i=1}^{D-2} \prod_{k=-\infty}^{\infty} e^{i2\pi k n_k^i \tau_1}\,. \tag{3.40}$$

Expanding the logarithm, the result is:

$$\mathcal{F} = \sum_{r=1}^{\infty} \frac{1}{4\sqrt{2}\pi}\int_0^{\infty}\frac{d\tau_2}{\tau_2^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, e^{-\frac{r^2\beta^2}{8\pi\tau_2}}\, e^{-2\pi\tau_2(D-2)\gamma_0(m)} \prod_{k=-\infty}^{\infty}\left(\frac{1}{1-e^{2\pi(-\tau_2\omega_k+i\tau_1 k)}}\right)^{D-2}, \tag{3.41}$$

where

$$\tau_2 = \frac{r\beta}{4\sqrt{2}\pi p^+}\,, \qquad \gamma_0\,(m) = \frac{1}{2}\sum_{k=-\infty}^{\infty}\omega_k = \frac{m}{2} + \sum_{k=1}^{\infty}\sqrt{k^2+m^2}\,. \tag{3.42}$$

Here $m = 2p^+\mu$ was defined in order to match up the notation of ref. [3] (see also ref. [7] for a detailed study about the zero point energy in general pp-waves backgrounds). This expression is the light-cone free energy for a pp-wave string gas, derived from second quantized string theory, just by evaluating second quantized operators in a pure state. Although this result is already known, the derivation of a string gas from a second quantized string theory is interesting by itself, since we have worked directly in the multi-string Hilbert space. With the TFD formulation of a free LCSFT at finite temperature, we hope to be able to introduce string interactions and derive a free energy for a non-ideal string gas. We will return to this point in the last section where a propose is presented. For now, we are ready to deal with a necessary ingredient to study LCSFT at finite temperature: the thermal propagator.

JHEP10(2005)063

4. The thermal propagator

In order to compute amplitudes in light-cone string theory at finite temperature, it is necessary to calculate the light-cone thermal propagator. The main goal of this section is to derive the light-cone thermal propagator for pp-wave closed string from the TFD formulation of LCSFT at finite temperature discussed early. For the flat space limit the thermal propagator can be expressed in terms of Theta functions, exactly as it was done in the zero temperature limit [35 − 37]. Owing to the modular properties of the Theta functions, this result can be very useful to study the dynamics of the theory at the Hagedorn limit.

In general, the real-time propagator for a field $\Phi(x)$ at finite temperature has the following matrix structure [38]:

$$G_\Phi\left(x-y\right) \rightarrow \begin{pmatrix} G_\Phi^{11}\left(x-y\right) & G_\Phi^{12}\left(x-y\right) \\ G_\Phi^{21}\left(x-y\right) & G_\Phi^{22}\left(x-y\right) \end{pmatrix}. \tag{4.1}$$

The G_Φ^{11} component is the usual physical propagator and it is this object that we are interested in. The other three propagators are considered as auxiliary ones (unphysical). They are useful to eliminate undesirable divergences coming from delta function products [38]. In the Schwinger's closed-time path formalism this structure is a consequence of the time path, that goes from $-\infty$ to $+\infty$ then back to $-\infty$ [39, 41]. In the TFD approach this matrix structure comes directly from the doubling of the field variables. For a general field Φ, the TFD propagators can be easily derived in a canonical way, by evaluating the following expectation value:

$$\begin{aligned} G_{TFD}^{11}(x-y) &= \left(0\left(\beta\right)\right| T\Phi(x)\Phi(y)\left|0\left(\beta\right)\right), \\ G_{TFD}^{12}(x-y) &= \left(0\left(\beta\right)\right| T\Phi(x)\widetilde{\Phi}(y)\left|0\left(\beta\right)\right), \\ G_{TFD}^{21}(x-y) &= \left(0\left(\beta\right)\right| T\widetilde{\Phi}(x)\Phi(y)\left|0\left(\beta\right)\right), \\ G_{TFD}^{22}(x-y) &= \left(0\left(\beta\right)\right| T\widetilde{\Phi}(x)\widetilde{\Phi}(y)\left|0\left(\beta\right)\right), \end{aligned} \tag{4.2}$$

where T means time ordering and $\left|0\left(\beta\right)\right)$ is the thermal vacuum defined for this field. The G_Φ^{11} component of the TFD propagator is the same as the Schwinger's propagator, while the auxiliary ones are related by means of a time path deformation [42].

Let's go back to the closed string theory. The pp-wave thermal physical propagator in the light-cone configuration space is defined by:

$$G\left(x^i(\sigma)-y^i(\sigma),\Delta x^-,\Delta x^+\right)_\beta = \left\langle 0\left(\beta\right)\right| \Phi(x^i(\sigma),x_0^-,x^+)\Phi(y^i(\sigma),y_0^-,y^+)\left|0\left(\beta\right)\right\rangle, \tag{4.3}$$

where $\Delta x^- = x_0^- - y_0^-$ and it is assumed that the light-cone time interval Δx^+ is always positive. In this expression it is necessary to impose the level matching condition in the thermal expectation value. We will come back to this point later.

In order to calculate the propagator, we are going to use the solution of the Schrödinger equation in configuration space, given by (2.13) and (2.14). Using the expansion (2.13), the commutation relations defined in (2.11) and the inverse of the thermal Bogoliubov transformation, the expectation value (4.3) can be written in terms of the modes y_k^i, x_k^i:

$$
\begin{aligned}
G\left(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+\right)_\beta = & \sum_{n_k^i=0}^{\infty} \int \frac{dp^+}{2\pi} e^{-i(\Delta x^- p^+ + \Delta x^+ p^-)} \times \\
& \times \prod_{i=1}^{D-2} \prod_{k=-\infty}^{\infty} \sqrt{\frac{\omega_k}{\pi}} \frac{H_{n_k^i}(\sqrt{\omega_k} x_k^i) H_{n_k^i}(\sqrt{\omega_k} y_k^i)}{2^{n_k^i}(n_k^i!)} \times \\
& \times e^{-\frac{1}{2}\omega_k\left((x_k^i)^2+(y_k^i)^2\right)} + \sum_{n_k^i=0}^{\infty} \int \frac{dp^+}{\pi} e^{-i(\Delta x^- p^+ + \Delta x^+ p^-)} \times \\
& \times \prod_{i=1}^{D-2} \prod_{k=-\infty}^{\infty} \sqrt{\frac{\omega_k}{\pi}} \frac{H_{n_k^i}(\sqrt{\omega_k} x_k^i) H_{n_k^i}(\sqrt{\omega_k} y_k^i)}{2^{n_k^i}(n_k^i!)} \times \\
& \times e^{-\frac{1}{2}\omega_k\left((x_k^i)^2+(y_k^i)^2\right)} \sinh^2(\theta_{n_k^i}) \,.
\end{aligned} \tag{4.4}
$$

As usual in real-time formalisms, the propagator is a sum of a zero temperature part (that is equal to the zero temperature propagator) plus finite temperature corrections. By using the thermal state condition (3.18) it is easy to see that this propagator satisfies the KMS condition:

$$
G\left(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+\right)_\beta = G\left(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+ + i\beta\right)_\beta \,. \tag{4.5}
$$

Now the expression for the propagator will be simplified. The first part is exactly the zero temperature contribution, so let's begin with the temperature corrections, that sit in the second part. By defining the parameter W_{lk}

$$
W_{k\,l} = e^{i\pi\omega_k \tau_l} = q_l^{\omega_k} \,, \qquad \tau_l = \frac{1}{2\pi p^+}\left(-\Delta x^+ + i\frac{\beta l}{\sqrt{2}}\right) , \tag{4.6}
$$

with $q_l = e^{i\pi\tau_l}$, and writing the propagator as

$$
\begin{aligned}
G\left(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+\right)_\beta = & \, G\left(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+\right)_{T=0} + \\
& + \Delta\left(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+\right)_\beta \,,
\end{aligned} \tag{4.7}
$$

the thermal corrections are given by the expression:

$$
\begin{aligned}
\Delta\left(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+\right)_\beta = & \sum_{l=1}^{\infty} \sum_{n_k^i=0}^{\infty} \int \frac{dp^+}{\pi} e^{-ip^+(\Delta x^- - i\frac{\beta l}{\sqrt{2}})} e^{-i\frac{A}{2p^+}(\Delta x^+ - i\frac{\beta l}{\sqrt{2}})} \\
& \times \prod_{i=1}^{D-2} \prod_{k=-\infty}^{\infty} \sqrt{\frac{\omega_k}{\pi}} \frac{H_{n_k^i}(\sqrt{\omega_k} x_k^i) H_{n_k^i}(\sqrt{\omega_k} y_k^i)}{2^{n_k^i}(n_k^i!)} \times \\
& \times W_{kl}^{n_k^i} e^{-\frac{1}{2}\omega_k\left((x_k^i)^2+(y_k^i)^2\right)} \,,
\end{aligned} \tag{4.8}
$$

For each value of i, k and l, the sum over n_k^i can be evaluated using the following identity for Hermite Polynomials [51, 52]

$$
\sum_{n_k^i} \frac{H_{n_k^i}(\sqrt{\omega_k} x_k^i) H_{n_k^i}(\sqrt{\omega_k} y_k^i)}{2^{n_k^i}(n_k^i!)} W_{kl}^{n_k^i} = \frac{1}{\sqrt{1-W_{kl}^2}} e^{\omega_k\left(\frac{2x_k^i y_k^i W_{kl}^2 - \left((x_k^i)^2+(y_k^i)^2\right)W_{kl}^2}{1-W_{kl}^2}\right)} , \tag{4.9}
$$

which gives the result:

$$
\begin{aligned}
\Delta\left(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+\right)_\beta &= \sum_{l=1}^{\infty}\int\frac{dp^+}{\pi}e^{-ip^+(\Delta x^- - i\frac{\beta l}{\sqrt{2}})}\times \\
&\quad\times\prod_{i=1}^{D-2}\prod_{k=-\infty}^{\infty}\left\{\left[\frac{\omega_k\, q_l^{\omega_k}}{\pi(1-q_l^{2\omega_k})}\right]^{\frac{1}{2}}\times\right. \qquad (4.10)\\
&\qquad\left.\times\, e^{\frac{\omega_k}{1-q_l^{2\omega_k}}\left\{2x_k^i y_k^i q_l^{\omega_k}-\left[\left(x_k^i\right)^2+\left(y_k^i\right)^2\right]\left(1+q_l^{2\omega_k}\right)\right\}}\right\} \\
&= \sum_{l=1}^{\infty}\int\frac{dp^+}{\pi}e^{-ip^+(\Delta x^- - i\frac{\beta l}{\sqrt{2}})}\times \\
&\quad\times\prod_{i=1}^{D-2}\prod_{k=-\infty}^{\infty}\left\{\left[\frac{i\omega_k}{2\pi\sin(\pi\tau_l\omega_k)}\right]^{\frac{1}{2}}\times\right. \qquad (4.11)\\
&\qquad\left.\times\, e^{\left\{\frac{i\omega_k}{2\sin(\pi\tau_l\omega_k)}\left[2x_k^i y_k^i-\left((x_k^i)^2+(y_k^i)^2\right)\cos(\pi\tau_l\omega_k)\right]\right\}}\right\}.
\end{aligned}
$$

This is a very nice result, since this expression has the same structure as the zero temperature propagator [53] (see also [54]). The only difference is the sum over l and the dependence on τ defined in (4.6) instead of just Δx^+. After making the same manipulation in the zero temperature part, the expression for the real-time thermal light-cone propagator for closed pp-wave strings is:

$$
\begin{aligned}
G(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+)_\beta &= \int\frac{dp^+}{2\pi}e^{-ip^+\Delta x^-}G(0,0,0,\Delta x^+)\times \\
&\quad\times\prod_{i=1}^{D-2}\prod_{k=-\infty}^{\infty}e^{\left\{\frac{\omega_k}{2i\sin(\frac{\Delta x^+\omega_k}{2p^+})}\left[2x_k^i y_k^i-\left((x_k^i)^2+(y_k^i)^2\right)\cos(\frac{\Delta x^+\omega_k}{2p^+})\right]\right\}}+ \\
&\quad+\sum_{l=1}^{\infty}\int\frac{dp^+}{\pi}e^{-ip^+(\Delta x^- - i\frac{\beta l}{\sqrt{2}})}G(0,0,0,\tau_l)\times \qquad (4.12)\\
&\quad\times\prod_{i=1}^{D-2}\prod_{k=-\infty}^{\infty}e^{\left\{\frac{i\omega_k}{2\sin(\pi\tau_l\omega_k)}\left[2x_k^i y_k^i-\left((x_k^i)^2+(y_k^i)^2\right)\cos(\pi\tau_l\omega_k)\right]\right\}},
\end{aligned}
$$

with

$$
G(0,0,0,\tau_l)=\prod_{k=-\infty}^{\infty}\left[\frac{\omega_k\, q_l^{\omega_k}}{\pi\left(1-q_l^{2\omega_k}\right)}\right]^{\frac{(D-2)}{2}}=\prod_{k=-\infty}^{\infty}\left[\frac{i\omega_k}{2\pi\sin(\pi\tau_l\omega_k)}\right]^{\frac{(D-2)}{2}}, \qquad (4.13)
$$

and $G(0,0,0,\Delta x^+) = G(0,0,0,\tau_{l=0})$. The expression for the propagator (4.12) can be written in terms of modular functions, at least for the flat space limit. Consider the temperature dependent part written in terms of q_l. Replacing the expression for the Fourier

JHEP10(2005)063

coefficients x_k^i and y_k^i, it can be presented as

$$
\begin{aligned}
&\Delta\left(\{x_k^i\},\{y_k^i\},\Delta x^-,\Delta x^+\right)_\beta = \\
&= \sum_{l=1}^{\infty}\int\frac{dp^+}{\pi}e^{-ip^+(\Delta x^- - i\frac{\beta l}{\sqrt{2}})}G(0,0,0,\tau_l)\times \qquad (4.14)\\
&\times\prod_{i=1}^{D-2} e^{\sum_{k=0}^{\infty}\left(\frac{\sqrt{2}}{2}\right)^2\int\frac{d\sigma}{2\pi p^+}\frac{d\sigma'}{2\pi p^+}\left\{\frac{\omega_k}{1-q_l^{2\omega_k}}\{2x^i(\sigma)y^i(\sigma')q_l^{\omega_k}-[(x^i(\sigma))^2+(y^i(\sigma'))^2](1+q_l^{2\omega_k})\}\cos(\frac{k(\sigma-\sigma')}{2p^+})\right\}}\,.
\end{aligned}
$$

In the flat space limit, $\omega_k = |k|$, the terms in the exponent are manipulated using the following properties of Theta functions [55, 35, 36]:[2]

$$\frac{\Theta_4'(z,q)}{\Theta_4(z,q)} = 4\sum_{k=1}^{\infty}\frac{q^k\sin(2kz)}{1-q^{2k}}\,, \qquad (4.15)$$

where the prime denotes differentiation with regard to z, and $\Theta_4(z,q)$ is defined as usual:

$$\Theta_4(z,q) = \sum_{k=-\infty}^{+\infty}(-)^k q^{k^2}e^{2kiz}\,. \qquad (4.16)$$

Considering $z = \frac{\sigma-\sigma'}{4p^+}$, and the following results from (4.15)

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{2kq^k\cos(2kz)}{1-q^{2k}} &= \frac{1}{4}\frac{d}{dz}\frac{\Theta_4'(z,q)}{\Theta_4(z,q)}\,, \qquad (4.17)\\
\sum_{k=0}^{\infty}\frac{k(1+q^{2k})}{1-q^{2k}}\cos(2kz) &= \frac{1}{8}\frac{d}{dz}\frac{\Theta_1'(z,q)}{\Theta_1(z,q)}\,, \qquad (4.18)
\end{aligned}
$$

the thermal closed string propagator can be elegantly expressed as:

$$
\begin{aligned}
G^{flat}(x^i(\sigma),y^i(\sigma'),\Delta x^-,\Delta x^+)_\beta = &\int\frac{dp^+}{2\pi}e^{-ip^+\Delta x^-}G(0,0,0,\Delta x^+)\times\\
&\times\prod_{i=1}^{D-2}\exp\left[\frac{1}{8}\int\frac{d\sigma d\sigma'}{(2\pi p^+)^2}\left(x^i(\sigma)y^i(\sigma')\frac{d}{dz}\frac{\Theta_4'(z,q')}{\Theta_4(z,q')}-\right.\right.\\
&\qquad -\frac{1}{2}(x^i(\sigma)x^i(\sigma')+y^i(\sigma)y^i(\sigma'))\times\\
&\qquad \left.\left.\times\frac{d}{dz}\frac{\Theta_1'(z,q)}{\Theta_1(z,q)}\right)\right]+\\
&+\sum_{l=1}^{\infty}\int\frac{dp^+}{\pi}e^{-ip^+\Delta x^-}G(0,0,0,\tau_l)\times\\
&\times\prod_{i=1}^{D-2}\exp\left[\frac{1}{8}\int\frac{d\sigma d\sigma'}{(2\pi p^+)^2}\left(x^i(\sigma)y^i(\sigma')\frac{d}{dz}\frac{\Theta_4'(z,q_l)}{\Theta_4(z,q_l)}-\right.\right.\\
&\qquad -\frac{1}{2}(x^i(\sigma)x^i(\sigma')+y^i(\sigma)y^i(\sigma'))\times\\
&\qquad \left.\left.\times\frac{d}{dz}\frac{\Theta_1'(z,q_l)}{\Theta_1(z,q_l)}\right)\right]\,, \qquad (4.19)
\end{aligned}
$$

[2] The expression (4.15) only works if $|Im(z)| < \frac{1}{2}Im(\pi\tau)$, that is the case.

where $q = q_{l=0}$. The Theta functions appearing here reflect the doubly periodicity of this propagator. In fact, the zero temperature part can be given in terms of Green functions on a torus [53]. The temperature corrections have the same doubly periodicity with a typical winding. Note that it was not needed to talk about time compactifications to get this winding. This is an effect of the Bogoliubov transformation and we are going to talk more about this in the next section. On the other hand, it will be interesting to give a path integral representation for this propagator in terms of a Schwinger's closed-time path and see how the winding appears in this scenario.

Now it must be emphasized that the propagators given at equations (4.12) and (4.19) are not really the physical propagators because they do not satisfy the level matching conditions. It is straightforward to take into account the constraint (2.10) just because this propagator has the same structure of zero temperature one. In general, for the zero temperature propagator, the constraint (2.10) can be improved using the Lagrange multiplier λ as follows

$$\begin{aligned} G^{\mathrm{phys.}}\left(X(\sigma), Y(\sigma')\right) &= \int d\lambda \, \langle 0| \, \Phi\left(X(\sigma)\right) e^{\sum_i \sum_k 2\pi\lambda k N_n^i} \Phi\left(Y(\sigma')\right) |0\rangle \\ &= \int d\lambda \, \langle 0| \, \Phi\left(X(\sigma)\right) \Phi\left(Y(\sigma' + 2\pi\lambda)\right) |0\rangle \, . \end{aligned} \tag{4.20}$$

As the Bogoliubov generator commutes with the first quantized operator $\sum_n n N_n^i$, the same manipulation can be done for the thermal propagator. So, to get the physical propagator, it is necessary just to replace $Y(\sigma')$ in (4.19) by $Y(\sigma' + 2\pi\lambda)$ and integrate over λ [36]. We can do the same in the pp-wave propagator (4.12) after an inverse Fourier transformation.

5. Conclusions and discussions

In this work a finite temperature formulation of pp-wave light-cone string theory was done using TFD; a real-time canonical approach. The main motivation of such an endeavor is to understand the role of superstring interactions in the Hagedorn behaviour. This is the core of a research program that has just started with this paper. The construction of a finite temperature string field allows us to understand the true dominant string objects at high temperatures. For example, it was pointed out in [2] that at finite string coupling, multi-string bound states may become dominant at high energies, differently from the case with zero string coupling, when the high energy behaviour is effectively that of a single string. The LCSFT at finite temperature is suitable to investigate this point, which is very important to really understand what the Hagedorn temperature means. In addiction, LCSFT at finite temperature provides a well suited way to show how the dictionary defined in [10] works at finite temperature.

In this paper all the TFD ingredients were developed in free light-cone string field. The free energy of an ideal light-cone pp-wave string gas was derived. Although the expression for this free energy is already known [2–6, 9, 56], it is the first time that a light-cone string gas is derived from a second quantized string theory, which is the natural scenario to study multi-string effects in light-cone. In addition, the TFD formulation of LCSFT allowed to derive a real-time thermal light-cone string propagator, which can be very useful to understand the dynamics of the theory at finite temperature. At this point we would like to make some comments on the results given in (3.41) and (4.12). The string gas free energy is usually derived by evaluating the partition function on a torus, defined in a target space with time coordinate compactified in a circle of radius beta, which originates a winding sum [57]. In general, this winding provides a tachyonic spectrum, which in some papers is interpreted as the source of the Hagedorn divergence. Here, it is necessary to emphasize that in the TFD approach, it is neither needed to talk about torus nor explicit time compactification. However, there is somehow a time periodicity and a torus structure behind the thermal vacuum defined in (3.19). This is clear looking at the expression (4.19) for the thermal propagator, that has the torus doubly periodicity and a typical winding sum. To understand this interesting point, let's go back to the first quantized string in TFD approach. The transverse partition function for a single string in light-cone is

$$z_{lc} = \mathrm{Tr}\left[e^{-\beta h + 2\pi i \lambda p}\right], \tag{5.1}$$

where λ is a Lagrange multiplier that imposes the level matching condition, h is the first quantized hamiltonian and p is the momentum operator that generates translations in the world sheet σ coordinate. This expression defines a torus with moduli space parameters defined by $\tau = \lambda + i\frac{\beta}{2\pi}$.

The same thermodynamic results derived from this partition function can be obtained in TFD approach, just making the same manipulations made here in the first quantized context. In this case, the tilde system is interpreted as an auxiliary string (tilde string) propagating backwards in euclidean time [56], and the first quantized thermal vacuum has a clear and beautiful topological interpretation. It was shown in [9, 56], that the first quantized thermal vacuum is a boundary state for the gluing of the string and tilde string, in order to make a torus with the same moduli space defined by (5.1). This gluing is the entanglement produced by the Bogoliubov transformation, that confines the field in a restricted region of the time axis. In fact, a generalized Bogoliubov transformation has been used to describe compactified bosonic and fermionic fields [58, 59]. As shown in section 3, in the second quantized scenario we have already started with a first quantized string and tilde string. This is clear in the expansions (3.9). In some ways, the effect of the second quantized Bogoluibov transformation is to produce a torus with these two first quantized strings plus a time compactification. The key to understand this effect lies on the thermal state condition (3.18). In the first quantized case, the thermal state condition is precisely the boundary equation for the gluing of the string and tilde string. This suggests that the second quantized thermal state condition may play a role of a boundary state for the gluing of two strings, propagating in a target space with compactified time coordinate. This point needs to be better investigated and will help to understand the TFD structure in second quantized string theory. Besides this, the flat space thermal propagator derived here is written in terms of Theta functions plus a typical winding sum. It will be interesting to show that this propagator can be derived from the torus Green equations, taking into account the KMS boundary conditions.

Finally, let's just write a few lines about how the interactions can be introduced. The main characteristic of the TFD approach, which really makes it suitable to study string interactions at finite temperature, is the use of a canonical quantum mechanical perturbation theory to calculate the free energy of an interacting gas. Following Umezawa [42], if the interaction hamiltonian is defined by H_I, one can define the parameter s by

$$H(s) = H_0 + sH_I\,, \tag{5.2}$$

where H_0 is the free hamiltonian. In the Heisenberg picture, the free energy for this interaction system is:

$$F = F(0) + \int_0^1 ds\,(0(\theta, s)|\,H_I\,|0(\theta, s))\,, \tag{5.3}$$

where $F(0)$ is the ideal gas free energy derived here and $|0(\theta, s))$ is the thermal vacuum defined for the hamiltonian $H(s)$. This expression shows that the variation of the free energy due to an interaction is proportional to the interaction thermal energy. There is a similar expression where the expectation value is taken over the thermal vacuum $|0(\theta))$ in the interaction representation:

$$F = F(0) + \int_0^1 ds\,(0(\theta)|\left[TH_I(t_0)\exp\left(-i\int_{\infty}^{\infty} dts\hat{H}_I(t)\right)\right]|0(\theta))\,, \tag{5.4}$$

where $\hat{H}_I(t)$ is $H_I - \widetilde{H}_I$ in an arbitrary time t. So, with the string interaction hamiltonian constructed in [23–26], it is possible to define a vacuum $|0(\theta, s))$ or an interaction representation as in (5.4), and to calculate the non ideal string free energy perturbatively. This further development of TFD in LCSFT context is a work in progress.

Acknowledgments

M.C.B.A. was partially supported by the CNPq Grant 302019/2003-0, A.L.G. and D.L.N. are supported by FAPESP post-doc fellowships.

References

[1] J.L.F. Barbon and E. Rabinovici, *Touring the hagedorn ridge*, `hep-th/0407236`.

[2] B.R. Greene, K. Schalm and G. Shiu, *On the hagedorn behaviour of pp-wave strings and $N = 4$ SYM theory at finite r-charge density*, *Nucl. Phys.* **B 652** (2003) 105 [`hep-th/0208163`].

[3] L.A. Pando Zayas and D. Vaman, *Strings in RR plane wave background at finite temperature*, *Phys. Rev.* **D 67** (2003) 106006 [`hep-th/0208066`].

[4] G. Grignani, M. Orselli, G.W. Semenoff and D. Trancanelli, *The superstring hagedorn temperature in a pp-wave background*, *JHEP* **06** (2003) 006 [`hep-th/0301186`].

[5] Y. Sugawara, *Thermal amplitudes in DLCQ superstrings on pp-waves*, *Nucl. Phys.* **B 650** (2003) 75 [`hep-th/0209145`].

[6] S.-j. Hyun, J.-D. Park and S.-H. Yi, *Thermodynamic behavior of IIA string theory on a pp-wave*, *JHEP* **11** (2003) 006 [hep-th/0304239].

[7] F. Bigazzi and A.L. Cotrone, *On zero-point energy, stability and hagedorn behavior of type-IIB strings on pp-waves*, *JHEP* **08** (2003) 052 [hep-th/0306102].

[8] D.L. Nedel, M.C.B. Abdalla and A.L. Gadelha, *Superstring in a pp-wave background at finite temperature: TFD approach*, *Phys. Lett.* **B 598** (2004) 121 [hep-th/0405258].

[9] M.C.B. Abdalla, A.L. Gadelha and D.L. Nedel, *Perspectives of TFD on string theory*, *Proc. Sci.* **WC2004** (2004) 020 [hep-th/0412134].

[10] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [hep-th/0202021].

[11] J.C. Plefka, *Lectures on the plane-wave string/gauge theory duality*, *Fortschr. Phys.* **52** (2004) 264 [hep-th/0307101].

[12] D. Sadri and M.M. Sheikh-Jabbari, *The plane-wave/super Yang-Mills duality*, *Rev. Mod. Phys.* **76** (2004) 853 [hep-th/0310119].

[13] R.C. Brower, D.A. Lowe and C.-I. Tan, *Hagedorn transition for strings on pp-waves and tori with chemical potentials*, *Nucl. Phys.* **B 652** (2003) 127 [hep-th/0211201].

[14] B. Sundborg, *The hagedorn transition, deconfinement and $N = 4$ SYM theory*, *Nucl. Phys.* **B 573** (2000) 349 [hep-th/9908001].

[15] N. Deo, S. Jain and C.-I. Tan, *Strings at high-energy densities and complex temperature*, *Phys. Lett.* **B 220** (1989) 125.

[16] N. Deo, S. Jain and C.-I. Tan, *String statistical mechanics above hagedorn energy density*, *Phys. Rev.* **D 40** (1989) 2626.

[17] N. Deo, S. Jain, O. Narayan and C.-I. Tan, *The effect of topology on the thermodynamic limit for a string gas*, *Phys. Rev.* **D 45** (1992) 3641.

[18] D.A. Lowe and L. Thorlacius, *Hot string soup*, *Phys. Rev.* **D 51** (1995) 665 [hep-th/9408134].

[19] M. Kaku and K. Kikkawa, *The field theory of relativistic strings, pt. 1: trees*, *Phys. Rev.* **D 10** (1974) 1110.

[20] M. Kaku and K. Kikkawa, *The field theory of relativistic strings, pt. 2: loops and pomerons*, *Phys. Rev.* **D 10** (1974) 1823.

[21] E. Cremmer and J.-L. Gervais, *Infinite component field theory of interacting relativistic strings and dual theory*, *Nucl. Phys.* **B 90** (1975) 410.

[22] M.B. Green, J.H. Schwarz and L. Brink, *Superfield theory of type-II superstrings*, *Nucl. Phys.* **B 219** (1983) 437.

[23] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background*, *Phys. Rev.* **D 66** (2002) 086004 [hep-th/0204146].

[24] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background, II*, *JHEP* **01** (2003) 036 [hep-th/0206073].

[25] A. Pankiewicz, *More comments on superstring interactions in the pp-wave background*, *JHEP* **09** (2002) 056 [hep-th/0208209].

[26] A. Pankiewicz and J. Stefanski, B., *pp-wave light-cone superstring field theory*, *Nucl. Phys.* **B 657** (2003) 79 [hep-th/0210246].

[27] Y. Takahasi and H. Umezawa, *Thermo field dynamics*, *Collect. Phenom.* **2** (1975) 55.

[28] A. Iorio, G. Lambiase and G. Vitiello, *Entangled quantum fields near the event horizon and entropy*, *Ann. Phys. (NY)* **309** (2004) 151.

[29] I. Ojima, *Gauge fields at finite temperatures: 'thermo field dynamics', Kms condition and their extension to gauge theories*, *Ann. Phys. (NY)* **137** (1981) 1.

[30] Y. Leblanc, *String field theory at finite temperature*, *Phys. Rev.* **D 36** (1987) 1780.

[31] R. Haag, *Local quantum physics: fields, particles, algebras*, Springer-Verlag, New York 1992.

[32] G.G. Emch, *Alebraic methods in statiscal and quantum field theory*, John Wiley, New York 1972.

[33] S.D. Mathur, *Is the polyakov path integral prescription too restrictive?*, hep-th/9306090.

[34] M. Spradlin and A. Volovich, *Light-cone string field theory in a plane wave*, hep-th/0310033.

[35] F. Jimenez Lorenzo, J. Ramirez Mittelbrunn, M. Ramon Medrano and G. Sierra, *Quantum mechanical amplitude for string propagation*, *Phys. Lett.* **B 171** (1986) 369, erratum *ibid.* **182** (1986) 414.

[36] L. Mezincescu, R.I. Nepomechie and P.K. Townsend, *Elliptic functions and the closed spinning string propagator*, *Class. and Quant. Grav.* **6** (1989) L29.

[37] J. Polchinski. *String theory, vol. II*, Cambridge University Press, Cambridge 1998.

[38] N.P. Landsman and C.G. van Weert, *Real and imaginary time field theory at finite temperature and density*, *Phys. Rept.* **145** (1987) 141.

[39] J.S. Schwinger, *Brownian motion of a quantum oscillator*, *J. Math. Phys.* **2** (1961) 407.

[40] L.V. Keldysh, *Diagram technique for nonequilibrium processes*, *Zh. Eksp. Teor. Fiz.* **47** (1964) 1515 [*Sov. Phys. JETP* **20** (1965) 1018].

[41] R.J. Rivers, *Path integral methods in quantum field theory*, Cambridge University Press, Cambridge 1987.

[42] H. Umezawa, *Advanced field theory: micro, macro and thermal field*, American Institute of Physics, New York 1993.

[43] H. Umezawa and Y. Yamanaka, *Micro, macro and thermal concepts in quantum field theory*, *Adv. Phys.* **37** (1988) 531.

[44] P. Elmfors and H. Umezawa, *Generalizations of the thermal Bogolyubov transformation*, *Physica* **A202** (1994) 577 [hep-th/9304089].

[45] P.A. Henning, *Thermo field dynamics for quantum fields with continuous mass spectrum*, *Phys. Rept.* **253** (1995) 235.

[46] M.C.B. Abdalla, A.L. Gadelha and I.V. Vancea, *On the* SU(1, 1) *thermal group of bosonic strings and D-branes*, *Phys. Rev.* **D 66** (2002) 065005 [hep-th/0203222].

[47] M.C.B. Abdalla, A.L. Gadelha and I.V. Vancea, *Bosonic Dp-branes at finite temperature in TFD approach*, *Nucl. Phys.* **127** *(Proc. Suppl.)* (2004) 92.

[48] M.C.B. Abdalla and A.L. Gadelha, *General unitary* $\mathrm{SU}(1,1)$ *TFD formulation*, *Phys. Lett.* **A 322** (2004) 31 [hep-th/0309254].

[49] M.C.B. Abdalla, A.L. Gadelha and D.L. Nedel, *On the entropy operator for the general* $\mathrm{SU}(1,1)$ *TFD formulation*, *Phys. Lett.* **A 334** (2005) 123 [hep-th/0409116].

[50] M.C.B. Abdalla, A.L. Gadelha and D.L. Nedel, *General unitary TFD formulation for superstrings*, *Proc. Sci.* **WC2004** (2004) 032 [hep-th/0412128].

[51] S.D. Mathur, A. Saxena and Y.K. Srivastava, *Scalar propagator in the pp-wave geometry obtained from* $AdS_5 \times S^5$, *Nucl. Phys.* **B 640** (2002) 367 [hep-th/0205136].

[52] H.S. Bateman, *Higher transcendental functions, vol. II*, McGraw-Hill, New York 1953.

[53] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory, vol. 2: loop amplitudes, anomalies and phenomenology*, Cambridge University Press, Cambridge 1987.

[54] C.-S. Chu and K. Kyritsis, *The string light cone in the pp-wave background*, *Phys. Lett.* **B 566** (2003) 240 [hep-th/0304191].

[55] E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Cambridge University Press, Cambridge 1969.

[56] M.C.B. Abdalla, A.L. Gadelha and D.L. Nedel, *Closed string thermal torus from thermofield dynamics*, *Phys. Lett.* **B 613** (2005) 213 [hep-th/0410068].

[57] J. Polchinski, *Evaluation of the one loop string path integral*, *Commun. Math. Phys.* **104** (1986) 37.

[58] J.C. da Silva, F.C. Khanna, A. Matos Neto and A.E. Santana, *Generalized bogoliubov transformation for confined fields: applications in casimir effect*, *Phys. Rev.* **A 66** (2002) 052101 [hep-th/0208183].

[59] H. Queiroz, J.C. da Silva, F.C. Khanna, J.M.C. Malbouisson, M. Revzen and A.E. Santana, *Thermofield dynamics and Casimir effect for fermions*, *Ann. Phys. (NY)* **317** (2005) 220.

JHEP10(2005)063